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Nonlocal Conserved Quantities, Balance Laws, and Equations of Motion

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A formalism is developed whereby balance laws are directly obtained from nonlocal (integrodifferential) linear second-order equations of motion for systems described by several dependent variables. These laws augment the equations of motion as further useful information about the physical system and, under certain conditions, are shown to reduce to conservation laws. The formalism can be applied to physical systems whose equations of motion may be relativistic and either classical or quantum. It is shown to facilitate obtaining global conservation laws for quantities which include energy and momentum. Applications of the formalism are given for a nonlocal Schrödinger equation and for a system of local relativistic equations of motion describing particles of arbitrary integral spin.

1. INTRODUCTION

Contemporary physicists try to understand nature at a fundamental level through at least two justifiable beliefs. One of these may be characterized by the "belief of constancy through change," and finds its expression in "conservation laws." The other may be characterized by the "belief in dynamical evolution," and finds its expression in "equations of motion" (sometimes referred to as "laws of motion"). [Further comments can be found in Appendix B, note 1; designated {B1}].

A third approach to understanding nature at a basic level [particularly in continuum mechanics; see, e.g., Scipio (1967)] is by means of "balance laws." Such laws are in some sense a connection between conservation laws (to which they sometimes reduce) and laws of motion (from which balance laws are sometimes derived). Their relevant differential (point) form (which may incorporate *integrals*) is schematized here, for an appropriate

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coordinate system (to make connection with usual nonrelativistic notation), as

$$\frac{\partial W}{\partial t}+\nabla \cdot T=R$$

where t is the time, W is the density of some property, T (a "flow" term) and R (a "source" term) are also related to this property, $\nabla \cdot$ is a generalized divergence (excluding time) of T. (W, T, and R may be tensors.) The integral law of balance for a body B may then be written as

$$\int_B \frac{\partial W}{\partial t}\,dV=-\int_B \nabla \cdot T\,dV+\int_B R\,dV$$

where dV is a generalized volume element.

The former equation will be referred to as the point form of a nonlocal (local) law or "equation of balance" if it does (does not) contain integrals. When the R term is absent, it will be referred to as a nonlocal (or local) "continuity equation" or law. The latter type of equation involving integrals which may include the entire system (over B, for example, if all fields are confined to the body) will always be referred to as a "global" law or equation. Finally, an equation showing the ordinary time derivative of a quantity to be zero relative to a particular coordinate system is called a "conservation law," the point form of that law being nonlocal if the quantity contains integrals (e.g., over other parts of B) {B2}.

In order to utilize the variational calculus related to a physical system's equations of motion (or imbed the equations of motion in a variational statement {B3}), an appropriate Lagrangian must be found. This is usually done by "educated guessing," although fairly general methods are available. For *local* laws of motion (i.e., laws described by differential equations) this imbedding can be seen in electromagnetic theory (Gelfand and Fomin, 1963), diffusion theory (Biot, 1970), nonrelativistic mechanics and special relativistic dynamics (Lanczos 1970), and quantum field theory (Byron and Fuller 1969, Chapter 2).

For *nonlocal* laws of motion (usually described by integrodifferential equations) an attempt to imbed the equation in a variational statement is generally even more difficult {B4}. Such imbedding is (as for local laws) important for a number of reasons, one of which is to facilitate computation of conserved quantities. Furthermore, nonlocal laws pertain to a large class of physical systems. Examples can be found in elasticity theory {B5}, radiative transport theory (Edelen, 1973), thermodynamics of fluids {B6}, electromagnetic theory {B7}, Fokker-type interactions; ({B8}; Havas, 1973), quantum mechanics {B9}, and quantum field theory [see the early paper by Ebel (1954)] {B10}. [A copious list of references is given in {B11}.]

The book by Edelen (1969*a*) (hereafter referred to as NV) based on an earlier series of papers (Edelen, 1969*b*) {B12} solves the nonlocal variational imbedding problem for a rather general Lagrangian. It also describes general relationships of conserved quantities to arbitrary Lagrangians by means of "energy-momentum complexes" arising from an extension of Noether's theorem (Noether, 1918; Edelen, 1971*a*–*d*) {B13}.

However, it would be interesting to find a more direct connection between nonlocal equations of motion and their balance laws, continuity equations, or conservation laws. It would be worthwhile, in particular, if one could avoid the Lagrangian altogether {B14}, thus obviating its construction (which may be difficult or time-consuming). Furthermore, a direct method could also clearly display a structural interconnectedness, thereby giving rise to the possibility of modifying existing (or adding new) interaction terms to the equations of motion and immediately seeing the consequences for the associated balance laws, continuity equations, or conservation laws. This may be of special value for linear approximations to nonlinear equations of motion (such as the Boltzmann equation).

Although such a connection has been made for some instances of the local case pertaining to, for example, linear second-order differential equations (referred to here as LSDEs) (Lurie *et al.*, 1966; Greider, 1984), the wider nonlocal case of linear second-order integrodifferential equations (referred to here as LSIDEs) does not appear to have been treated.

The main results of this paper appear as theorems showing how to write down nonlocal balance laws, and give conditions under which nonlocal continuity equations and nonlocal conservation laws obtain, by considering the LSIDEs (or adjoint LSIDEs) *alone* (i.e., without having to explicitly introduce a Lagrangian {B15}). Details can be found in Gould (1982) {B16}.

The paper is organized as follows:

Section 2 describes the general way in which the "main results" were obtained.

The main results are then found in Section 3. These appear as three theorems showing how to directly relate nonlocal systems of equations of motion to corresponding nonlocal balance laws, continuity equations, or conservation laws.

Two applications of the formalism are given in Section 4. One is for a nonlocal Schrödinger equation. The other is for a system of local relativistic equations describing particles of arbitrary integral spin.

Section 5 discusses the main results, including comments concerning the physical significance of energy-momentum complexes for LSIDE systems. It also describes the importance of boundary conditions.

Appendix A sketches how nonlocal equations of motion, the energy-momentum complex, and balance laws arise out of the nonlocal calculus

of variations. This is done to give the reader some idea of the "scaffolding" that went into constructing the paper's main results.

Finally, Appendix B elucidates certain remarks made in the text.

2. SKETCH OF THE MAIN RESULTS

The formalism of this paper is based on the observation that by utilizing the variational calculus it is possible to imbed certain classes of integrodifferential equations in a variational formulation through the use of a suitable Lagrangian (or Lagrangian density, in general) $\mathscr{L}$. This is true, in particular, if the class considered is any system of linear second-order integrodifferential equations (LSIDEs) having an appropriate continuity structure.

It is then possible to utilize a nonlocal extension of Noether's theorem (Edelen, 1971*a–d*) to arrive at the general formulation of the nonlocal energy-momentum complex (a quantity sometimes related to the energy-momentum density), W_k^m or $\Theta^{\alpha\beta}$, each of which depends on $\mathscr{L}$.

One can again refer to the Noether theorem in order to arrive at a formula which expresses nonlocal balance laws as a function of $\mathscr{L}$. The balance laws go over to nonlocal continuity equations if there are no sources (or "sinks") present. And the continuity equations can, in turn, become nonlocal conservation laws (e.g., when the fields vanish over the boundary). (See the Remarks following Theorem III for further discussion.)

However, the reader should be aware of the fact that no Lagrangian needs to be calculated in order to apply the formalism of this paper even though a Lagrangian was used to obtain the paper's main results (as explained in {B15}). The reason for this is that one is only interested in obtaining balance laws, continuity equations, or conservation laws for systems whose dependent (or "state") variables satisfy Euler's equations (viz., the LSIDEs "equations of motion"); and, as can be seen, the LSIDEs as well as other quantities in the paper's main results (Theorems I–III) do not contain any Lagrangian.

The connections established between the LSIDEs, LSDEs, and those other quantities may be schematized as follows:

LSIDEs $\dashrightarrow$	LSDEs
$\downarrow\uparrow$ using the formula for the nonlocal W_k^m or $\Theta^{\alpha\beta}$	$\downarrow\uparrow$ using the formula for the local W_k^m or $\Theta^{\alpha\beta}$
Nonlocal balance laws	Local balance laws
$\downarrow\uparrow$ ur LSIDEs	$\downarrow\uparrow$ ur LSDEs
Nonlocal continuity equations	Local continuity equations
$\downarrow\uparrow$ ur LSIDEs	$\downarrow\uparrow$ ur LSDEs
Nonlocal conservation laws	Local conservation laws

where $\dashrightarrow$ means "weakly leading to," ur means "leads to under certain restrictions on," and W_k^m or $\Theta^{\alpha\beta}$ are energy-momentum-complex functions.

3. CONNECTION BETWEEN NONLOCAL EQUATIONS OF MOTION, THE ENERGY-MOMENTUM COMPLEX, AND BALANCE LAWS

As a result of introducing a certain general Lagrangian $\mathscr{L}$ into the Euler equations, one arrives at a general expression for both the LSIDEs and their adjoint LSIDEs. Then, introducing $\mathscr{L}$ into the general expression for the momentum-energy complex (obtained from a nonlocal generalization of Noether's theorem), one also arrives at the form of what will be called a "canonical energy-momentum complex" corresponding to the LSIDEs. (These procedures are indicated in Appendix A.)

The first of the main results is then obtained explicitly as follows.

Theorem I. Given any system of LSIDEs

$$a^{tr}_{\Sigma\Lambda}\partial_t\partial_r\phi^\Lambda + a^{r}_{\Sigma\Lambda}\partial_r\phi^\Lambda + a_{\Sigma\Lambda}\phi^\Lambda + \int_{D^*} K_{\Sigma\Lambda}(x, z)\phi^\Lambda(z)\, dV(z)$$

$$+ \int_{D^*} H^{q}_{\Sigma\Lambda}(x, z)\partial_q\phi^\Lambda(z)\, dV(z) = f_\Sigma \tag{3.1}$$

describing a physical system, where t, r go from 1 to n=the number of independent variables, each capital Greek letter is to stand for σ indices, $\{\phi^\Lambda\}$ is an N-tuple with N being the number of dependent variables, and the adjoint system of LSIDEs with N-tuple $\{\psi^\Gamma\}$ is given by

$$\partial_q\partial_t(a^{qt}_{\Gamma\Sigma}\psi^\Gamma) - \partial_t(a^{t}_{\Gamma\Sigma}\psi^\Gamma) + a_{\Gamma\Sigma}\psi^\Gamma + \int_{D^*} K_{\Gamma\Sigma}(z, x)\psi^\Gamma(z)\, dV(z)$$

$$- \int_{D^*} \psi^\Gamma(z)\frac{\partial H^{t}_{\Gamma\Sigma}(z, x)}{\partial x^t}\, dV(z) = -g_\Sigma \tag{3.2}$$

Then a canonical energy-momentum complex corresponding to equations (3.1) is

$$W^m_k = {}^{\mathrm{L}}W^m_k + {}^{\mathrm{NL}}W^m_k \tag{3.3}$$

where $k, m = 1, \ldots, n$; the *local* part of W^m_k is

$$\begin{aligned}
{}^{\mathrm{L}}W^m_k := {} & -\tfrac{1}{2}[a^m_{\Gamma\Lambda} - (\partial_q a^{qm}_{\Gamma\Lambda})]\psi^\Gamma\partial_k\phi^\Lambda + \tfrac{1}{2}[a^m_{\Gamma\Lambda} - (\partial_q a^{qm}_{\Gamma\Lambda})]\phi^\Lambda\partial_k\psi^\Gamma \\
& + a^{mq}_{\Gamma\Lambda}(\partial_q\phi^\Lambda)(\partial_k\psi^\Gamma) + a^{qm}_{\Gamma\Lambda}(\partial_q\psi^\Gamma)(\partial_k\phi^\Lambda) \\
& - \delta^m_k\{f_\Gamma\psi^\Gamma - g_\Lambda\phi^\Lambda - [a_{\Gamma\Lambda} - \tfrac{1}{2}(\partial_q a^{q}_{\Gamma\Lambda}) + \tfrac{1}{2}(\partial_q\partial_r a^{qr}_{\Gamma\Lambda})]\psi^\Gamma\phi^\Lambda \\
& - \tfrac{1}{2}[a^{r}_{\Gamma\Lambda} - (\partial_q a^{qr}_{\Gamma\Lambda})](\partial_r\phi^\Lambda)\psi^r + \tfrac{1}{2}[a^{r}_{\Gamma\Lambda} - (\partial_q a^{qr}_{\Gamma\Lambda})](\partial_r\psi^\Gamma)\phi^\Lambda \\
& + a^{qr}_{\Gamma\Lambda}(\partial_q\psi^\Gamma)(\partial_r\phi^\Lambda)\}
\end{aligned} \tag{3.3a}$$

and the *nonlocal* part of W_k^m is

$$^{\mathrm{NL}}W_k^m := -[\partial_k \phi^\Lambda(x)]k_\Lambda^m(x;\psi^\Gamma) + \delta_k^m\{\phi^\Lambda(x)k_{2\Lambda}(x;\psi^\Gamma) + [\partial_q\phi^\Lambda(x)]k_\Lambda^q(x;\psi^\Gamma) + \psi^\Gamma(x)k_{1\Gamma}(x;\phi^\Lambda)\} \tag{3.3b}$$

wherein

$$k_{2\Lambda}(x;\psi^\Gamma) := \int_{D^*} K_{\Gamma\Lambda}(z,x)\psi^\Gamma(z)\,dV(z) \tag{3.4a}$$

$$k_{1\Gamma}(x;\phi^\Lambda) := \int_{D^*} [K_{\Gamma\Lambda}(x,z)\phi^\Lambda(z) + H_{\Gamma\Lambda}^q(x,z)\partial_q\phi^\Lambda(z)]\,dV(z) \tag{3.4b}$$

$$k_\Lambda^r(x;\psi^\Gamma) := \int_{D^*} H_{\Gamma\Lambda}^r(x,z)\psi^\Gamma(z)\,dV(z) \tag{3.4c}$$

Remarks.

1. D^* is a compact set in n-dimensional number space of the n independent variables x^k or z^k (designated collectively as x or z, respectively). A volume element of D^* at x is designated $dV(x)$.

2. The $\{\phi^\Lambda\}$ and $\{\psi^\Gamma\}$ are each an N-tuple of dependent variables and adjoint dependent variables, respectively. These N-tuples may then be combined and written as a (direct product) $2N$-tuple $\{\varphi^\Sigma\} := \{\phi^\Lambda, \psi^\Gamma\}$, where φ^Σ and its partials $\partial_i\varphi^\Sigma := \partial\varphi^\Sigma(x)/\partial x^i$ are continuous functions of the x^k (except when they are shown to depend on the z^k) and defined over a Banach space designated $\mathscr{D}_1(D^*; 2N)$.

3. Terms like $a_{\Sigma\Lambda}^{tr}$, $a_{\Sigma\Lambda}^t$, $a_{\Sigma\Lambda}$, f_Σ, and g_Σ are each functions of the x^k.

4. The usual summation convention (implied sum over repeated indices) holds throughout unless indicated otherwise.

5. It is important to realize that the LSIDEs (and other equations in this paper) are not necessarily covariant even though the notation suggests otherwise. The reason for this notation is to facilitate applications to relativity.

6. As an application of the index notation, if $\sigma = s$ for a spin-s field, then one may have ϕ^Λ as a tensor of rank σ. So if $n = 4$ for space-time, then $N = 4^s$ (however, symmetry conditions on the dependent variables can reduce the value of N).

7. It should be noted that no symmetry is assumed for such terms as the $a_{\Sigma\Lambda}^{tr}$, in contrast to the case where there are no Greek indices—i.e., for just one dependent variable (or for application to self-adjoint equations to which the above reduces in certain cases) where $a^{tr} = a^{rt}$. The general form given by equations (3.1) will be relevant for (Maxwell's and) higher-spin equations, as will be shown at the end of the Applications section.

Thus, *Theorem I yields a direct method of relating a system of nonlocal equations of motion to a corresponding energy-momentum complex.* For

example, if one can relate quantities in the adjoint LSIDEs to those in the original LSIDEs (as is possible in a number of cases), then the W_k^m may be directly written using equation (3.3) in terms of the original dependent variables $\{\phi^\Lambda\}$. Also, one could modify terms within the context of the original LSIDEs ("equations of motion") to consider different interactions and see the effect on the energy-momentum complex or vice versa.

Of course, how easy it is to get the W_k^m in terms of the $\{\phi^\Lambda\}$ depends on the complexity of the LSIDEs. But even the adjoint functions $\{\psi^\Gamma\}$ may have physical significance (as explained in Section 5).

3.1. A Symmetry-Oriented Energy-Momentum Complex

If one raises indices (assuming a nonsingular metric) in equation (3.3) to obtain $W^{\alpha\beta}$, as usual for the local cases in field theory, the resultant expression (as well, of course, as the former expression) is generally not symmetric in α and β. To obtain a quantity $\Theta^{\alpha\beta}$ which *is* symmetric in many cases it will be necessary to modify the expression (3.3), a virtue of symmetric expressions being that they manifestly imply conservation of "angular momentum" (as, e.g., the symmetric energy-momentum density tensor in electromagnetic theory) {B17}.

The goal of symmetrization is aided by generalizing the procedure used for local equations of motion (Soper, 1976, Section 9.4) as indicated in Section 3 of Appendix A. Thus one gets, using Theorem I, the following result.

Theorem II. Given any LSIDE system [equation (3.1)] along with its adjoint system [equation (3.2)] and canonical energy-momentum complex [equation (3.3)], then an associated *symmetry-oriented* energy-momentum complex is

$$\Theta^{\alpha\beta} = \mathscr{W}^{\alpha\beta} - \partial_\mu G^{\alpha\beta\mu} \tag{3.5}$$

where

$$\mathscr{W}^{\alpha\beta} := \tfrac{1}{2} W^{\alpha\beta} \tag{3.5a}$$

with $W^{\alpha\beta}$ obtained from equation (3.3), while

$$G^{\alpha\beta\mu} := \tfrac{1}{2}(S^{\beta\mu\alpha} + S^{\alpha\mu\beta} - S^{\alpha\beta\mu}) \tag{3.5b}$$

The quantities making up $G^{\alpha\beta\mu}$ are found from

$$S^\mu_{\alpha\beta} = {}^{\mathrm{L}}S^\mu_{\alpha\beta} + {}^{\mathrm{NL}}S^\mu_{\alpha\beta} \tag{3.6}$$

where the local part is

$$\begin{aligned} {}^{\mathrm{L}}S^\mu_{\alpha\beta} := \tfrac{1}{4}\{ &-\tfrac{1}{2}[a^\mu_{\Gamma\Lambda} - (\partial_q a^{q\mu}_{\Lambda\Gamma})]\psi^\Lambda + \tfrac{1}{2}[a^\mu_{\Gamma\Lambda} - (\partial_q a^{q\mu}_{\Gamma\Lambda})]\phi^\Lambda \\ &+ a^{r\mu}_{\Lambda\Gamma}(\partial_r \psi^\Lambda) + a^{\mu r}_{\Lambda\Gamma}(\partial_r \phi^\Lambda)\}[\tilde{M}_{\alpha\beta}]^\Gamma_\Omega(\phi^\Omega + \psi^\Omega) \end{aligned} \tag{3.6a}$$

and the nonlocal part is

$$^{\mathrm{NL}}S^{\mu}_{\alpha\beta} := -\tfrac{1}{8}\{k^{\mu}_{\Gamma}(x;\psi^{\Gamma})\}[\tilde{M}_{\alpha\beta}]^{\Gamma}_{\Omega}(\phi^{\Omega}+\psi^{\Omega}) \tag{3.6b}$$

(with Ω running over the N components of ϕ^{Ω} and of ψ^{Ω}) such that, for $\sigma = s+1$,

$$[\tilde{M}_{\alpha\beta}]^{\omega\mu_1\cdots\mu_s}_{\sigma\nu_1\cdots\nu_s} = (M_{\alpha\beta})^{\omega}_{\sigma}\delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2}\cdots\delta^{\mu_s}_{\nu_s} + (M_{\alpha\beta})^{\mu_1}_{\nu_1}\delta^{\omega}_{\sigma}\delta^{\mu_2}_{\nu_2}\cdots\delta^{\mu_s}_{\nu_s} + \cdots + (M_{\alpha\beta})^{\mu_s}_{\nu_s}\delta^{\omega}_{\sigma}\delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2}\cdots\delta^{\mu_{s-1}}_{\nu_{s-1}} \tag{3.7}$$

where

$$(M_{\alpha\beta})^{\xi}_{\zeta} = -\delta^{\xi}_{\alpha}\eta_{\beta\zeta} + \delta^{\xi}_{\beta}\eta_{\alpha\zeta} \tag{3.7a}$$

are the components of one of the six infinitesimal generating matrices $M_{\alpha\beta}(=-M_{\beta\alpha})$ of Lorentz transformations, and $\eta_{\mu\nu}$ is the Minkowski metric.

Remarks.

1. A proof of equation (3.7) is given in Appendix C of Gould (1982).

2. Antisymmetry: $M_{\alpha\beta} = -M_{\beta\alpha}$ leads to $S^{\mu}_{\alpha\beta} = -S^{\mu}_{\beta\alpha}$ and thus $G^{\alpha\beta\mu} = -G^{\alpha\mu\beta}$ [as explained following equation (A4.2) of Appendix A].

As for the previous theorem, one can *directly* relate an LSIDE system (the equations of motion) to a corresponding energy-momentum complex via Theorem II. (The usefulness of this theorem for a local system of equations of motion will be given in the second example of the Applications section.)

3.2. Nonlocal Balance Laws for LSIDE Systems

Using the preceding results, it is now possible to establish a direct connection between LSIDE systems and nonlocal balance laws (along with nonlocal conservation laws). This is indicated by Section A4 of Appendix A, which gives rise to the following result.

Theorem III. Given any LSIDE system (3.1) describing a physical system along with the adjoint LSIDE system (3.2), then an associated nonlocal balance law is

$$\begin{aligned}
\partial_m W^m_k = &-[\partial_k f_{\Gamma}(x)]\psi^{\Gamma} + [\partial_k g_{\Gamma}(x)]\phi^{\Gamma}(x) \\
&+ (\partial_k\{a_{\Gamma\Lambda}(x) - \tfrac{1}{2}[\partial_q a^q_{\Gamma\Lambda}(x)] + \tfrac{1}{2}[\partial_q\partial_r a^{qr}_{\Gamma\Lambda}(x)]\})\psi^{\Gamma}(x)\phi^{\Lambda}(x) \\
&+ \tfrac{1}{2}\{\partial_k[a^r_{\Gamma\Lambda}(x) - \partial_q a^{qr}_{\Gamma\Lambda}(x)]\}[\partial_r\phi^{\Lambda}(x)]\psi^{\Gamma}(x) \\
&- \tfrac{1}{2}\{\partial_k[a^r_{\Gamma\Lambda}(x) - \partial_q a^{qr}_{\Gamma\Lambda}(x)]\}[\partial_r\psi^{\Gamma}(x)]\phi^{\Lambda}(x) \\
&- [\partial_k a^{qr}_{\Gamma\Lambda}(x)][\partial_q\psi^{\Gamma}(x)]\partial_r\phi^{\Lambda}(x) \\
&+ \phi^{\Lambda}(x)\frac{\partial k_{2\Lambda}(x;\psi^{\Gamma})}{\partial x^k} + \psi^{\Gamma}(x)\frac{\partial k_{1\Gamma}(x;\phi^{\Lambda})}{\partial x^k} \\
&+ \partial_q\phi^{\Lambda}(x)\frac{\partial k^q_{\Lambda}(x;\psi^{\Gamma})}{\partial x^k}
\end{aligned} \tag{3.8}$$

with the W_k^m given by equations (3.3). In addition there results the nonlocal balance law

$$\partial_\beta \Theta^{\alpha\beta} = \partial_\beta \mathscr{W}^{\alpha\beta} \tag{3.9}$$

with $\Theta^{\alpha\beta}$ given by equation (3.5) and $\mathscr{W}^{\alpha\beta}$ related to the W_k^m via definition (3.5a). If the rhs of equation (3.8) or (3.9) is zero, there result nonlocal continuity equations. And if, further, the time derivatives $\partial_0 W_k^m = 0$ or $\partial_0 \Theta^{\alpha 0} = 0$, then nonlocal conservation laws result.

Remarks.

1. The following terminology is introduced to expand upon the last two sentences: If $\partial_m W_k^m$ (or $\partial_\beta \Theta^{\alpha\beta}$) is given by equation (3.8) [or equation (3.9)] in a particular reference frame with the rhs nonzero, then the equations will be referred to as the point form of *nonlocal* (*local*) *balance laws* if they contain (do not contain) integrals. If the rhs of equation (3.8) [or equation (3.9)] is zero in a particular reference frame, then the equations will be referred as the point form of *nonlocal* (*local*) *continuity equations* if they contain (do not contain) integrals. Finally, if in a particular reference frame

$$0 = \partial_m W_k^m = \partial_0 W_k^0 \qquad (\text{for } m \neq 0) \tag{3.10}$$

or

$$0 = \partial_\beta \Theta^{\alpha\beta} = \partial_0 \Theta^{\alpha 0} \qquad (\text{for } \beta \neq 0) \tag{3.11}$$

the equations will be referred to as the point form of *nonlocal (local) conservation laws* if they contain (do not contain) integrals; the subscript 0 is used instead of, say, the $m = 1$ or $\beta = 1$ coordinate, so that ∂_0 stands for the derivative with respect to time.

2. Although the energy-momentum (or symmetry-oriented energy-momentum) complex can sometimes be directly associated with energy or momentum for particle dynamics, in field theory it is usually necessary to integrate those quantities over some region (say, 3-dimensional space) in order to obtain energy or momentum. Consequently, the integrated equations (3.8) [or (3.9)] will be referred to as *global balance laws.* In case the integrated equations result in the integral of $\partial_0 W_k^0$ or $\partial_0 \Theta^{\alpha 0}$ vanishing over some spatial hypervolume there will result *global conservation laws.* One may have, in particular, the interesting case of global conservation laws obtaining from nonlocal balance laws, the conservation laws being valid only for the system as a whole.

3. The nonlocal energy-momentum complex W_k^m arises out of applying a one-parameter family of coordinate transformations (of class C^u in the independent variables x^k) and function variations to the nonlocal action (see Chapter 4 of NV). This is a starting point for obtaining Noether's theorem. However, a special case of that theorem results when one obtains

the derivatives of the energy-momentum complex for those systems where $\partial_m W_k^m = 0$, a conservation law being obtained when the partial derivative (with respect to time) of the W_k^m is zero, while the more general situation which gives rise to balance laws is obtained when the rhs of the previous equation (instead of being zero) is equal to terms depending on the Lagrangian under conditions where the Euler equations are satisfied [cf. equation (2.136) from Theorem 2.11 of NV and equation (3.8) above]. The rhs is then associated with "source quantities," while the W_k^m may be referred to as components of the "flux vector." These results (i.e., balance laws stemming from Noether's theorem) should not be confused with "transversality conditions" (pertaining to restrictions that coordinate and function variations have *on the boundary* of D^*) even though both results involve the W_k^m {B18}.

Thus, *Theorem III* (along with Theorems I and II) *enables one to relate nonlocal systems of equations of motion (LSIDE systems) directly to corresponding nonlocal balance laws, continuity equations, or conservation laws.*

There are different ways of applying the formalism. For example, as will be shown in a local case of the Applications section, in order to try to obtain the W_k^m or $\Theta^{\alpha\beta}$ in terms of the ϕ^Λ alone (rather than with the adjoint functions in addition) the procedure will be to identify quantities in the adjoint LSIDE system with those in the original LSIDE system (the "equations of motion"). Whenever such a procedure is carried out, it will be referred to a establishing *compatability conditions* between the LSIDE systems.

4. APPLICATIONS

In this section two examples are given to show precisely how the formalism can be applied. The first example comes from nonlocal nonrelativistic quantum mechanics and the second example comes from local relativistic quantum field theory.

4.1. A Nonlocal Case

An example of a nonlocal "equation of motion" frequently used in contemporary physics is the nonlocal Schrödinger equation [{B19}, {B20}]

$$(-\hbar^2/2M)\nabla^2\phi - i\hbar\partial_4\phi + \int_{D^*} U(x, z)\phi(z)\, dV(z) = 0 \tag{4.1}$$

where, in general, D^* stands for the 4-dimensional region with $x \equiv (x^1, x^2, x^3, x^4)$ or $z \equiv (z^1, z^2, z^3, z^4)$, $U(x, z)$ is a nonlocal potential because of the z dependence, ∂_4 is the partial derivative with respect to the time

coordinate x^4, ϕ is the wave function, $i=\sqrt{-1}$, $\hbar$ = Planck's constant/2π, and M is the particle's mass. Starting with this equation, application of the formalism is fairly easy.

Begin with Theorem I (but for no Greek indices). The basic equation (3.1) is

$$a^{tr}\partial_t\partial_r\phi + a^r\partial_r\phi + a\phi + \int_{D^*} K(x, z)\phi(z)\, dV(z)$$

$$+ \int_{D^*} H^q(x, z)\partial_q\phi(z)\, dV(z) = f \tag{4.2}$$

Comparing Eqs. (4.1) and (4.2) gives

$$a^{tr} = -\hbar^2/2M \quad (t=r=1,2,3), \qquad a^{tr}\equiv 0 \quad (t\neq r), \qquad a^{44}\equiv 0, \qquad f\equiv 0$$

$$a^1 = a^2 = a^3 \equiv 0, \qquad a^4 = -i\hbar, \qquad a\equiv 0, \tag{4.3}$$

$$K(x, z) = U(x, z), \qquad H^q(x, z)\equiv 0$$

The basic adjoint equation (3.2) is

$$a^{qt}\partial_q\partial_t\psi - a^t\partial_t\psi + a\psi + \int_{D^*} K(z, x)\psi(z)\, dV(z)$$

$$- \int_{D^*} \frac{\partial H^t(z, x)}{\partial x^t}\psi(z)\, dV(z) = -g \tag{4.4}$$

which, with equations (4.3), gives

$$(-\hbar^2/2M)\nabla^2\psi + i\hbar\partial_4\psi + \int_{D^*} U(z, x)\psi(z)\, dV(z) = -g \tag{4.5}$$

Now taking the complex conjugate of equation (4.5) gives

$$(-\hbar^2/2M)\nabla^2\bar{\psi} - i\hbar\partial_4\bar{\psi} + \int_{D^*} \bar{U}(z, x)\bar{\psi}(z)\, dV(z) = -g \tag{4.6}$$

where the overbar on a quantity means complex conjugate. A "compatibility" of equations (4.1) and (4.6) holds for

$$\bar{\psi} = \phi, \qquad \int_{D^*} \bar{U}(z, x)\bar{\psi}(z)\, dV(z) = \int_{D^*} U(x, z)\phi(z)\, dV(z), \qquad g\equiv 0 \tag{4.7}$$

The second equation of (4.7) is satisfied for

$$\bar{U}(z, x) = U(x, z) \tag{4.8}$$

which is the requirement that the Hamiltonian

$$H := (-\hbar^2/2M)\nabla^2\phi + \int_{D^*} U(x, z)\phi(z)\, dV(z) \tag{4.9}$$

be Hermitian {B21}.

Finishing the application of Theorem I to equation (4.1) is quite simple: Using equations (4.3) and (4.7), equations (3.4a)-(3.4c) become

$$k_2(x; \bar{\phi}) = \int_{D^*} U(z, x)\bar{\phi}(z)\, dV(z) \tag{4.10}$$

$$k_1(x; \phi) = \int_{D^*} U(x, z)\phi(z)\, dV(z) \tag{4.11}$$

$$k^r(x; \phi) \equiv 0 \tag{4.12}$$

4.1.1. Computing the W_k^m

When equations (4.3), (4.7), and (4.10)-(4.12) are put into equation (3.3) (again remembering there are no Greek indices) there immediately results

$$\begin{aligned} W_k^m = {} & a^{mq}(\partial_q\phi)(\partial_k\bar{\phi}) + a^{qm}(\partial_q\bar{\phi})(\partial_k\phi) - \tfrac{1}{2}a^m\bar{\phi}\partial_k\phi + \tfrac{1}{2}a^m\phi\partial_k\bar{\phi} \\ & + \delta_k^m\{\tfrac{1}{2}a^4(\partial_4\phi)\bar{\phi} - \tfrac{1}{2}a^4(\partial_4\bar{\phi})\phi - a^{qr}(\partial_q\bar{\phi})(\partial_r\phi) \\ & + \phi(x)k_2(x; \bar{\phi}) + \bar{\phi}(x)k_1(x; \phi)\} \end{aligned} \tag{4.13}$$

for the energy-momentum complex.

Explicit calculation of the W_k^m is obtained using equation (4.13) along with (4.3) and (4.10)-(4.12):

$$\begin{aligned} W_4^4 = {} & (\hbar^2/2M)(\nabla\phi)\cdot(\nabla\bar{\phi}) + \phi(x)\int_{D^*} U(z, x)\bar{\phi}(z)\, dV(z) \\ & + \bar{\phi}(x)\int_{D^*} U(x, z)\phi(z)\, dV(z) \end{aligned} \tag{4.14}$$

for the nonlocal energy density {B22}.

Using the more suggestive vector notation we can write W_k^4 $(k = 1, 2, 3)$ as

$$-\mathbf{W}^4 = \tfrac{1}{2}\bar{\phi}(-i\hbar\nabla\phi) + \tfrac{1}{2}\phi(i\hbar\nabla\bar{\phi}) = \Re[\bar{\phi}(-i\hbar\nabla)\phi] = \Re[\bar{\phi}P\phi] \tag{4.15}$$

where $\mathbf{W}^4 := (W_1^4, W_2^4, W_3^4)$, $\Re$ means the real part, and $P := -i\hbar\nabla$ is the momentum operator. Thus, $\mathbf{W}^4$ essentially represents the momentum-density components of the field (the same expression as for the usual local Schrödinger equation).

For $k = m = 1, 2, 3$ one has (no sum on m)

$$W_m^m = (-\hbar^2/M)|\partial_m\phi|^2 - \tfrac{1}{2}ih(\bar{\phi}\partial_4\phi - \phi\partial_4\bar{\phi}) + W_4^4 \tag{4.16}$$

where $|\partial_m\phi|^2 := (\partial_m\phi)(\partial_m\bar{\phi})$ and W_4^4 is given by equation (4.14).

For $k = 4$ and $m = 1, 2, 3$ there results

$$\mathbf{W}_4 = (-\hbar^2/M)\Re[(\nabla\phi)(\partial_4\bar{\phi})] \tag{4.17}$$

where $\mathbf{W}_4 := (W_4^1, W_4^2, W_4^3)$.

For $k \neq m = 1, 2, 3$ there results

$$W_k^m = (-\hbar^2/M)\Re[(\partial_m\phi)(\partial_k\bar{\phi})] \qquad \text{for} \quad k = 1, 2, 3 \tag{4.18}$$

All 16 components of W_k^m can be symbolized:

$$(\mathbf{W}_4, W_4^4) \quad \text{and} \quad (\mathbf{W}_k, W_k^4) \quad \text{for } k = 1, 2, 3 \tag{4.19}$$

where $\mathbf{W}_q := (W_q^1, W_q^2, W_q^3)$ for $q = 1, 2, 3, 4$. Thus, regarding the $\{W_k^m\}$ of expression (4.19) as making up a 4×4 matrix, the nonlocal terms only occur along the main diagonal. The off-diagonal elements give the same expressions as for the local Schrödinger equation. To use a continuum mechanics analogy: it is as if "normal stresses" were due to the nonlocality but "shears" were not. (For the local Schrödinger equation the diagonal terms, in addition, are the usual ones.)

4.1.2. Nonlocal Balance Laws for Energy and Momentum Densities

When equations (4.3), (4.7), and (4.10)–(4.12) are substituted into equations (3.8) of Theorem III (with no Greek indices) there results

$$\partial_m W_k^m = 2\Re\left(\phi(x)\frac{\partial k_2(x;\bar{\phi})}{\partial x^k}\right) \tag{4.20}$$

Then, using equations (4.19), these equations can be separated into two groups.

The first is

$$[\nabla \cdot \mathbf{W}_k + \partial_4 W_k^4]\mathbf{e}^k = 2\Re[\phi(x)\,\nabla k_2(x;\bar{\phi})] \tag{4.21}$$

where $\mathbf{e}^k$ is a unit vector along the x^k axis and $k = 1, 2, 3$. These equations represent the nonlocal balance law for momentum density.

The second group is

$$\nabla \cdot \mathbf{W}_4 + \partial_4 W_4^4 = 2\Re[\phi(x)\,\partial_4 k_2(x;\bar{\phi})] \tag{4.22}$$

and represents the nonlocal balance law for energy density.

One sees from equation (4.10) that the x-dependence is contained only in $U(z, x)$, the "kernel" of $k_2(x, \bar{\phi})$. Therefore, if any component of the

gradient operator $\nabla := \mathbf{e}^1\partial_1 + \mathbf{e}^2\partial_2 + \mathbf{e}^3\partial_3$ gives zero when operating on $U(z, x)$, then equation (4.21) gives a nonlocal continuity equation for that component of the momentum density.

If $\partial U(z, x)/\partial x^4 = 0$, then the rhs of equation (4.22) is zero and one has a nonlocal continuity equation for the energy density. If, further, $\nabla \cdot \mathbf{W}_4 = 0$, then (retaining x^4 instead of x^0 for the time coordinate) one has

$$\partial_4 W_4^4 = 0 \tag{4.23}$$

a nonlocal conservation law for the energy density. By integrating equation (4.23) over some spatial subset d^* of D^*, there results

$$\int_{d^*} (\partial_4 W_4^4)\, dx^1\, dx^2\, dx^3 = d\left(\int_{d^*} W_4^4\, dx^1\, dx^2\, dx^3\right)\Big/ dx^4 = 0 \tag{4.23a}$$

[Less stringently, one may have integrated $\nabla \cdot \mathbf{W}_4$ over the 3-space, applied Gauss' theorem to d^* (or to D^*), and then found the integral of $\partial_4 W_4^4$ over d^* (or D^*) to have vanished.] This says that the energy within d^* remains constant over time. Notice that the energy within d^* is not determined only by what exists within that region, but *also* on what exists *outside* it (and within D^*), as can be seen from equation (4.14). This "global" aspect is an expected characteristic of starting with nonlocal equations of motion. It is *not* an expected situation(!) for local equations of motion [e.g., the usual Schrödinger equation has $W_4^4 = (\hbar^2/2M)(\nabla\phi)\cdot(\nabla\bar{\phi}) + U(x)\bar{\phi}(x)\phi(x)$].

As an interesting comparison, let us see what the balance laws are for the local Schrödinger equation. Theorem I easily yields the W_k^m symbolized by equations (4.19) for the local case. Theorem III (even more easily) yields that instead of equation (4.21) one has

$$[\nabla \cdot \mathbf{W}_k + \partial_4 W_k^4]\mathbf{e}^k = -\mathbf{F}\rho \tag{4.24}$$

and instead of equations (4.22) there results

$$\nabla \cdot \mathbf{W}_4 + \partial_4 W_4^4 = (\partial_4 U)\rho \tag{4.25}$$

where $\mathbf{F} = -\nabla U$ is the local force and $\rho = \phi\bar{\phi}$ is the local probability density (Havas, 1978 {B23}).

4.2. A Local Case

In order to describe the dynamics of massive particles of integral spin s in the presence of sources one may consider the fundamental equations of motion

$$\partial^\alpha U^{(q+1)}_{[\alpha\beta][\xi_1\xi_2][\xi_3\xi_4]\cdots[\xi_{2q-1}\xi_{2q}]\xi_{2q+1}\cdots\xi_{s+q-1}} + \chi^2 U^{(q)}_{\beta[\xi_1\xi_2][\xi_3\xi_4]\cdots[\xi_{2q-1}\xi_{2q}]\xi_{2q+1}\cdots\xi_{s+q-1}}$$
$$= J^{(q)}_{\beta[\xi_1\xi_2][\xi_3\xi_4]\cdots[\xi_{2q-1}\xi_{2q}]\xi_{2q+1}\cdots\xi_{s+q-1}} \tag{4.26}$$

where the $U^{\cdot}_{\cdots}$ represent real potentials, while χ^2 and $J^{\cdot}_{\cdots}$, respectively,

essentially stand for the particle mass and source density {B24} ([$\zeta\tau$] means $U^{\cdot\cdot\cdot}_{\cdots}$ is antisymmetric in ζ and τ.)

Using the definition

$$U^{(q+1)}_{[\alpha\beta][\xi_1\xi_2]\cdots[\xi_{2q-1}\xi_{2q}]\xi_{2q+1}\cdots\xi_{s+q-1}} := \partial_\alpha U^{(q)}_{\beta[\xi_1\xi_2]\cdots[\xi_{2q-1}\xi_{2q}]\xi_{2q+1}\cdots\xi_{s+q-1}} - \partial_\beta U^{(q)}_{\alpha[\xi_1\xi_2]\cdots[\xi_{2q-1}\xi_{2q}]\xi_{2q+1}\cdots\xi_{s+q-1}} \tag{4.27}$$

along with the Minkowski metric $\eta_{\alpha\beta}$ and its contravariant counterpart (so, e.g., $\partial^\alpha = \eta^{\alpha\sigma}\partial_\sigma$), one can write the system of equations (4.26) for $t := s+q-1$ as

$$\begin{aligned}
&\eta^{\alpha\sigma}\eta_{\beta\nu}\eta_{\xi_1\varepsilon_1}\cdots\eta_{\xi_t\varepsilon_t}\partial_\sigma\partial_\alpha U^{(q)\nu[\varepsilon_1\varepsilon_2]\cdots\varepsilon_t} \\
&-\eta^{\alpha\sigma}\eta_{\alpha\nu}\eta_{\xi_1\varepsilon_1}\eta_{\xi_2\varepsilon_2}\cdots\eta_{\xi_t\varepsilon_t}\partial_\sigma\partial_\beta U^{(q)\nu[\varepsilon_1\varepsilon_2]\cdots\varepsilon_t} \\
&+\chi^2\eta_{\beta\nu}\eta_{\xi_1\varepsilon_1}\eta_{\xi_2\varepsilon_2}\cdots\eta_{\xi_t\varepsilon_t}U^{(q)\nu[\varepsilon_1\varepsilon_2]\cdots\varepsilon_t} = J^{(q)}_{\beta[\xi_1\xi_2]\cdots\xi_t}
\end{aligned} \tag{4.28}$$

our starting equations of motion.

For the LSIDE system (3.1) containing only local terms one has the LSDE system

$$\begin{aligned}
&a^{\alpha\sigma}_{\beta\nu\xi_1\varepsilon_1\xi_2\varepsilon_2\cdots\xi_t\varepsilon_t}\partial_\alpha\partial_\sigma\phi^{(q)\nu\varepsilon_1\varepsilon_2\cdots\varepsilon_t} + a^{\alpha}_{\beta\nu\xi_1\varepsilon_1\xi_2\varepsilon_2\cdots\xi_t\varepsilon_t}\partial_\alpha\phi^{(q)\nu\varepsilon_1\varepsilon_2\cdots\varepsilon_t} \\
&+a_{\beta\nu\xi_1\varepsilon_1\xi_2\varepsilon_2\cdots\xi_t\varepsilon_t}\phi^{(q)\nu\varepsilon_1\varepsilon_2\cdots\varepsilon_t} = f^{(q)}_{\beta\xi_1\xi_2\cdots\xi_t}
\end{aligned} \tag{4.29}$$

Comparing equations (4.28) and (4.29), using

$$\begin{aligned}
&\eta^{\alpha\sigma}\eta_{\alpha\nu}\eta_{\xi_1\varepsilon_1}\eta_{\xi_2\varepsilon_2}\cdots\eta_{\xi_t\varepsilon_t}\partial_\sigma\partial_\beta U^{(q)\nu[\varepsilon_1\varepsilon_2]\cdots\varepsilon_t} \\
&= \delta^\alpha_\nu\delta^\sigma_\beta\eta_{\xi_1\varepsilon_1}\eta_{\xi_2\varepsilon_2}\cdots\eta_{\xi_t\varepsilon_t}\partial_\sigma\partial_\alpha U^{(q)\nu[\varepsilon_1\varepsilon_2]\cdots\varepsilon_t}
\end{aligned}$$

and realizing that the $U^{(q)\nu[\varepsilon_1\varepsilon_2]\cdots\varepsilon_t}$ may be ordered in some way (e.g., $U^{(q)0[00]\cdots0}$, $U^{(q)1[00]\cdots0}$, ..., etc.), one can make the identifications

$$\begin{aligned}
a^{\alpha\sigma}_{\beta\nu\xi_1\varepsilon_1\xi_2\varepsilon_2\cdots\xi_t\varepsilon_t} &= \eta^{\alpha\sigma}\eta_{\beta\nu}\eta_{\xi_1\varepsilon_1}\eta_{\xi_2\varepsilon_2}\cdots\eta_{\xi_t\varepsilon_t} - \delta^\alpha_\nu\delta^\sigma_\beta\eta_{\xi_1\varepsilon_1}\eta_{\xi_2\varepsilon_2}\cdots\eta_{\xi_t\varepsilon_t} \\
a^{\alpha}_{\beta\nu\xi_1\varepsilon_1\xi_2\varepsilon_2\cdots\xi_t\varepsilon_t} &\equiv 0 \\
a_{\beta\nu\xi_1\varepsilon_1\xi_2\varepsilon_2\cdots\xi_t\varepsilon_t} &= \chi^2\eta_{\beta\nu}\eta_{\xi_1\varepsilon_1}\eta_{\xi_2\varepsilon_2}\cdots\eta_{\xi_t\varepsilon_t} \\
\phi^{(q)\nu\varepsilon_1\varepsilon_2\cdots\varepsilon_t} &= U^{(q)\nu[\varepsilon_1\varepsilon_2]\cdots\varepsilon_t} \\
f^{(q)}_{\beta\xi_1\xi_2\cdots\xi_t} &= J^{(q)}_{\beta[\xi_1\xi_2]\cdots\xi_t}
\end{aligned} \tag{4.30}$$

The adjoint equations for the LSIDE system will then contain only local terms and are obtained from equation (3.2) as

$$\begin{aligned}
&a^{\alpha\sigma}_{\nu\beta\xi_1\varepsilon_1\xi_2\varepsilon_2\cdots\xi_t\varepsilon_t}\partial_\alpha\partial_\sigma\psi^{(q)\nu\varepsilon_1\varepsilon_2\cdots\varepsilon_t} - a^{\alpha}_{\nu\beta\xi_1\varepsilon_1\xi_2\varepsilon_2\cdots\xi_t\varepsilon_t}\partial_\alpha\psi^{(q)\nu\varepsilon_1\varepsilon_2\cdots\varepsilon_t} \\
&+a_{\nu\beta\xi_1\varepsilon_1\xi_2\varepsilon_2\cdots\xi_t\varepsilon_t}\psi^{(q)\nu\varepsilon_1\varepsilon_2\cdots\varepsilon_t} = -g^{(q)}_{\beta\xi_1\xi_2\cdots\xi_t}
\end{aligned} \tag{4.31}$$

Compatibility of equations (4.29) and (4.31) can then be obtained using

$$\psi^{(q)\nu\varepsilon_1\varepsilon_2\cdots\varepsilon_t} = \phi^{(q)\nu\varepsilon_1\varepsilon_2\cdots\varepsilon_t} \tag{4.32}$$

$$-g^{(q)}_{\beta\xi_1\xi_2\cdots\xi_t} = f^{(q)}_{\beta\xi_1\xi_2\cdots\xi_t} \tag{4.33}$$

with $a^{\alpha}_{\nu\beta\xi_1\varepsilon_1\xi_2\varepsilon_2\cdots\xi_t\varepsilon_t} \equiv 0$, thereby requiring

$$a^{\alpha\sigma}_{\nu\beta\xi_1\varepsilon_1\xi_2\varepsilon_2\cdots\xi_t\varepsilon_t} = a^{\alpha\sigma}_{\beta\nu\xi_1\varepsilon_1\xi_2\varepsilon_2\cdots\xi_t\varepsilon_t} \tag{4.34}$$

Since one is only dealing with a local set of equations, then ${}^{\mathrm{NL}}W_k^m \equiv 0$ in equation (3.3b). So, putting $m \to \mu$, $k \to \kappa$, $q \to \rho$ (and $q \to \zeta$, $r \to \rho$ within $\{\cdot\}$) in equation (3.3a) and employing equations (4.30), (4.32), and (4.33) along with definition (3.5a) yields

$$\mathcal{W}_\kappa^\mu = a_{\Gamma\Lambda}^{\mu\rho}(\partial_\rho\phi^\Lambda)(\partial_\kappa\phi^\Gamma) - \delta_\kappa^\mu\{f_\Gamma\phi^\Gamma - \tfrac{1}{2}a_{\Gamma\Lambda}\phi^\Gamma\phi^\Lambda + \tfrac{1}{2}a_{\Gamma\Lambda}^{\zeta\rho}(\partial_\zeta\phi^\Gamma)(\partial_\rho\phi^\Lambda)\} \quad (4.35)$$

where use has been made of

$$a_{\Gamma\Lambda}^{\rho\mu}(\partial_\rho\psi^\Gamma)(\partial_\kappa\phi^\Lambda) = a_{\Lambda\Gamma}^{\rho\mu}(\partial_\rho\psi^\Lambda)(\partial_\kappa\phi^\Gamma) = a_{\Gamma\Lambda}^{\mu\rho}(\partial_\rho\phi^\Lambda)(\partial_\kappa\phi^\Gamma) \quad (4.35a)$$

The expressions on the rhs of (4.35) can now be obtained in terms of expressions from our starting equations.

From equation (4.30), with $\alpha \to \zeta$, $\sigma \to \rho$, $\beta \to \gamma$, and $\nu \to \lambda$, there results

$$\tfrac{1}{2}a_{\Gamma\Lambda}^{\zeta\rho}(\partial_\zeta\phi^\Gamma)(\partial_\rho\phi^\Lambda) = \tfrac{1}{4}U^{(q+1)}_{[\zeta\gamma][\xi_1\xi_2]\cdots\xi_t}U^{(q+1)[\zeta\gamma][\xi_1\xi_2]\cdots\xi_t} \quad (4.35b)$$

using definition (4.27). From equations (4.30), with $\beta \to \gamma$ and $\nu \to \lambda$, one gets

$$-\tfrac{1}{2}a_{\Gamma\Lambda}\phi^\Gamma\phi^\Lambda = -\tfrac{1}{2}\chi^2 U^{(q)\gamma[\xi_1\xi_2]\cdots\xi_t}U^{(q)}_{\gamma[\xi_1\xi_2]\cdots\xi_t} \quad (4.35c)$$

Finally, using equations (4.32) and (4.34) along with appropriate change of indices, it can be shown that

$$a_{\Gamma\Lambda}^{\rho\mu}(\partial_\rho\psi^\Gamma)(\partial_\kappa\phi^\Lambda) = a^{\mu\rho}_{\gamma\lambda\xi_1\varepsilon_1\xi_2\varepsilon_2\cdots\xi_t\varepsilon_t}\partial_\rho U^{(q)\lambda[\varepsilon_1\varepsilon_2]\cdots\varepsilon_t}\partial_\kappa U^{(q)\gamma[\xi_1\xi_2]\cdots\xi_t} \quad (4.35d)$$

To obtain the energy-momentum complex, put equation (4.35d) in (4.35a) and this result along with equations (4.35b) and (4.35c) into (4.35) to give, using (4.30) (with $\alpha \to \mu$, $\sigma \to \rho$, $\beta \to \gamma$, and $\nu \to \lambda$)

$$\begin{aligned}\mathcal{W}_\kappa^\mu = {} & \eta^{\mu\rho}\eta_{\gamma\lambda}\eta_{\xi_1\varepsilon_1}\eta_{\xi_2\varepsilon_2}\cdots\eta_{\xi_t\varepsilon_t}\partial_\rho U^{(q)\lambda[\varepsilon_1\varepsilon_2]\cdots\varepsilon_t}\partial_\kappa U^{(q)\gamma[\xi_1\xi_2]\cdots\xi_t} \\ & - \delta_\lambda^\mu\delta_\gamma^\rho\eta_{\xi_1\varepsilon_1}\eta_{\xi_2\varepsilon_2}\cdots\eta_{\xi_t\varepsilon_t}\partial_\rho U^{(q)\lambda[\xi_1\varepsilon_2]\cdots\varepsilon_t}\partial_\kappa U^{(q)\gamma[\xi_1\xi_2]\cdots\xi_t} \\ & - \delta_\kappa^\mu\{J^{(q)}_{\beta[\xi_1\xi_2]\cdots\xi_t}U^{(q)\beta[\xi_1\xi_2]\cdots\xi_t} \\ & + \tfrac{1}{4}U^{(q+1)}_{[\zeta\gamma][\xi_1\xi_2]\cdots\xi_t}U^{(q+1)[\zeta\gamma][\xi_1\xi_2]\cdots\xi_t} \\ & - \tfrac{1}{2}\chi^2 U^{(q)\gamma[\xi_1\xi_2]\cdots\xi_t}U^{(q)}_{\gamma[\xi_1\xi_2]\cdots\xi_t}\}\end{aligned} \quad (4.36)$$

As $\mathcal{W}^{\tau\mu} = \mathcal{W}_\kappa^\mu\eta^{\tau\kappa}$, then equation (4.36) becomes

$$\begin{aligned}\mathcal{W}^{\tau\mu} = {} & U^{(q+1)[\mu\lambda][\varepsilon_1\varepsilon_2]\cdots\varepsilon_t}\partial^\tau U^{(q)}_{\lambda[\varepsilon_1\varepsilon_2]\cdots\varepsilon_t} - \eta^{\tau\mu}\{J^{(q)}_{\beta[\xi_1\xi_2]\cdots\xi_t}U^{(q)\beta[\xi_1\xi_2]\cdots\xi_t} \\ & + \tfrac{1}{4}U^{(q+1)}_{[\omega\nu][\xi_1\xi_2]\cdots\xi_t}U^{(q+1)[\omega\nu][\xi_1\xi_2]\cdots\xi_t} \\ & - \tfrac{1}{2}\chi^2 U^{(q)}_{\lambda[\xi_1\xi_2]\cdots\xi_t}U^{(q)\lambda[\xi_1\xi_2]\cdots\xi_t}\}\end{aligned} \quad (4.37)$$

using equation (4.27).

Since ${}^{\mathrm{NL}}S_{\alpha\beta}^\mu \equiv 0$, then putting equation (3.6a) in (3.6) gives

$$\begin{aligned}S_{\alpha\beta}^\mu &= \tfrac{1}{4}\{a_{\Lambda\Gamma}^{r\mu}(\partial_r\phi^\Lambda) + a_{\Gamma\Lambda}^{\mu r}(\partial_r\phi^\Lambda)\}[\tilde{M}_{\alpha\beta}]^\Gamma_\Omega 2\phi^\Omega \\ &= a_{\Gamma\Lambda}^{\mu r}(\partial_r\phi^\Gamma)[\tilde{M}_{\alpha\beta}]^\Gamma_\Omega\phi^\Omega \\ &= a^{\mu\rho}_{\omega\lambda\mu_1\varepsilon_1\mu_2\varepsilon_2\cdots\mu_t\varepsilon_t}\partial_\rho U^{(q)\lambda[\varepsilon_1\varepsilon_2]\cdots\varepsilon_t}[\tilde{M}_{\alpha\beta}]^{\omega\mu_1\cdots\mu_t}_{\sigma\nu_1\cdots\nu_t}U^{(q)\sigma[\nu_1\nu_2]\cdots\nu_t} \\ &= -U^{(q+1)[\mu}{}_{\alpha]}{}^{[\nu_1\nu_2]\cdots\nu_t}U^{(q)}_{\beta[\nu_1\nu_2]\cdots\nu_t} \\ &\quad + U^{(q+1)[\mu}{}_{\beta]}{}^{[\nu_1\nu_2]\cdots\nu_t}U^{(q)}_{\alpha[\nu_1\nu_2]\cdots\nu_t} + \{S_{\alpha\beta}^\mu\}\end{aligned} \quad (4.38)$$

where the second step follows using (4.34) and the third follows from (4.30), (4.32), and (4.34) along with $\gamma\xi_1\xi_2\ldots\xi_t \to \omega\mu_1\mu_2\ldots\mu_t$. The last step in equation (4.38) comes about as follows: Put the expression (3.7a) into (3.7) and then multiply by $U^{(q)\sigma[\nu_1\nu_2]\cdots\nu_t}$. The first term from the rhs of (3.7) will, with (3.7a), give the relevant ones for the first two terms on the rhs of (4.38); the remaining terms are lumped into $\{s^{\mu}_{\alpha\beta}\}$.

Now raising indices in equation (4.38) gives

$$S^{\alpha\beta\mu} = -U^{(q+1)[\mu\alpha][\nu_1\nu_2]\cdots\nu_t}U^{(q)\beta}_{[\nu_1\nu_2]\cdots\nu_t} + U^{(q+1)[\mu\beta][\nu_1\nu_2]\cdots\nu_t}U^{(q)\alpha}_{[\nu_1\nu_2]\cdots\nu_t} + \{s^{\mu}_{\alpha\beta}\} \tag{4.38a}$$

Then, using (3.5b) gives

$$G^{\alpha\beta\mu} = U^{(q+1)[\beta\mu][\nu_1\nu_2]\cdots\nu_t}U^{(q)\alpha}_{[\nu_1\nu_2]\cdots\nu_t} + \tfrac{1}{2}\{g^{\alpha\beta\mu}\} \tag{4.38b}$$

where

$$g^{\alpha\beta\mu} := s^{\beta\mu\alpha} + s^{\alpha\mu\beta} - s^{\alpha\beta\mu} = -g^{\alpha\mu\beta}$$

and thus

$$\partial_\beta\partial_\mu g^{\alpha\beta\mu} \equiv 0 \tag{4.38c}$$

a fact which will be used later.

The symmetry-oriented energy-momentum complex can now be obtained by putting equations (4.37) and (4.38b) into (3.5) to get

$$\begin{aligned}\Theta^{\alpha\beta} = {} & U^{(q+1)[\beta\lambda][\varepsilon_1\varepsilon_2]\cdots\varepsilon_t}\partial^{\alpha}U^{(q)}_{\lambda[\varepsilon_1\varepsilon_2]\cdots\varepsilon_t} \\ & - \eta^{\alpha\beta}\{\cdots\} + U^{(q+1)[\mu\beta][\nu_1\nu_2]\cdots\nu_t}\partial_\mu U^{(q)\alpha}_{[\nu_1\nu_2]\cdots\nu_t} \\ & + \partial_\mu U^{(q+1)[\mu\beta][\nu_1\nu_2]\cdots\nu_t}U^{(q)\alpha}_{[\nu_1\nu_2]\cdots\nu_t} - \tfrac{1}{2}\partial_\mu\{g^{\alpha\beta\mu}\}\end{aligned} \tag{4.39}$$

where $\{\cdots\}$ comes from equation (4.37). Then the sum of the first and fourth terms is evaluated using definition (4.27) along with the antisymmetry of $U^{(q+1)[\beta\lambda]\cdots}$. And the third term is evaluated after multiplying (4.26) by $U^{(q)\cdots}$. When this is done equation (4.39) becomes

$$\begin{aligned}\Theta^{\alpha\beta} = {} & \theta^{\alpha\beta} + J^{(q)\beta[\nu_1\nu_2]\cdots\nu_t}U^{(q)\alpha}_{[\nu_1\nu_2]\cdots\nu_t} \\ & - \eta^{\alpha\beta}J^{(q)}_{\lambda[\xi_1\xi_2]\cdots\xi_t}U^{(q)\lambda[\xi_1\xi_2]\cdots\xi_t} - \tfrac{1}{2}\partial_\mu\{g^{\alpha\beta\mu}\}\end{aligned} \tag{4.40}$$

where $\theta^{\alpha\beta}$ is defined as

$$\begin{aligned}\theta^{\alpha\beta} := {} & U^{(q+1)[\alpha}{}_{\lambda][\varepsilon_1\varepsilon_2]\cdots\varepsilon_t}U^{(q+1)[\beta\lambda][\varepsilon_1\varepsilon_2]\cdots\varepsilon_t} \\ & - \chi^2 U^{(q)\alpha}_{[\nu_1\nu_2]\cdots\nu_t}U^{(q)\beta[\nu_1\nu_2]\cdots\nu_t} \\ & - \eta^{\alpha\beta}\{\tfrac{1}{4}U^{(q+1)}_{[\omega\nu][\xi_1\xi_2]\cdots\xi_t}U^{(q+1)[\omega\nu][\xi_1\xi_2]\cdots\xi_t} \\ & - \tfrac{1}{2}\chi^2 U^{(q)}_{\lambda[\xi_1\xi_2]\cdots\xi_t}U^{(q)\lambda[\xi_1\xi_2]\cdots\xi_t}\}\end{aligned} \tag{4.40a}$$

and is clearly *symmetric* in α and β.

The divergence Theorem III is now employed by substituting equations (4.30), (4.32), and (4.33) into the rhs of equation (3.8) to get (as k's $\equiv 0$)

$$\partial_\beta \Theta^{\alpha\beta} = -U^{(q)\beta[\xi_1\xi_2]\cdots\xi_t}\partial^\alpha J^{(q)}_{\beta[\xi_1\xi_2]\cdots\xi_t} \tag{4.41}$$

Also, from equation (4.40) along with (4.38c) there results, using definition (4.27),

$$\partial_\beta \Theta^{\alpha\beta} = \partial_\beta \theta^{\alpha\beta} - J^{(q)}_{\beta[\xi_1\xi_2]\cdots\xi_t} U^{(q+1)[\beta\alpha][\xi_1\xi_2]\cdots\xi_t} - \partial^\alpha J^{(q)}_{\beta[\xi_1\xi_2]\cdots\xi_t} U^{(q)\beta[\xi_1\xi_2]\cdots\xi_t}$$
$$+ \partial_\beta J^{(q)\beta[\nu_1\nu_2]\cdots\nu_t} U^{(q)\alpha}_{[\nu_1\nu_2]\cdots\nu_t} \tag{4.42}$$

From equations (4.41) and (4.42) follows

$$\partial_\beta \theta^{\alpha\beta} = J^{(q)}_{\beta[\xi_1\xi_2]\cdots\xi_t} U^{(q+1)[\beta\alpha][\xi_1\xi_2]\cdots\xi_t} - \partial_\beta J^{(q)\beta[\nu_1\nu_2]\cdots\nu_t} U^{(q)\alpha}_{[\nu_1\nu_2]\cdots\nu_t} \tag{4.43}$$

a *local balance law* with sources designated by the rhs.

In regions where there are no sources $J^{\cdots}_{\cdots} \equiv 0$ and (4.43) implies that

$$\partial_\beta \theta^{\alpha\beta} = 0 \tag{4.44}$$

which is a *local continuity equation* for the symmetric energy-momentum density tensor.

For regions where sources exist one may write equation (4.43) as

$$\partial_\beta(\theta^{\alpha\beta} + \mathcal{M}^{\alpha\beta}) = 0 \tag{4.45}$$

a local continuity equation for source density *plus* energy-momentum density tensor, where $\mathcal{M}^{\alpha\beta}$ is defined such that

$$\partial_\beta \mathcal{M}^{\alpha\beta} := -\{J^{(q)}_{\beta[\xi_1\xi_2]\cdots\xi_t} U^{(q+1)[\alpha\beta][\xi_1\xi_2]\cdots\xi_t} - \partial_\beta J^{(q)\beta[\nu_1\nu_2]\cdots\nu_t} U^{(q)\alpha}_{[\nu_1\nu_2]\cdots\nu_t}\}$$

Equations (4.44) and (4.45) may immediately imply local and global conservations laws [cf. equation (3.11) and definitions following] for the energy-momentum of the field and for the energy-momentum of the field-plus-source, respectively {B25}.

Although equations (4.44) and (4.45) are known results, it should be mentioned that when they were first introduced into the literature (e.g., Fierz, 1939) no Lagrangian was known and the energy-momentum density tensors were obtained by judicious guessing.

5. DISCUSSION

The main result of this paper has been to establish direct relations between linear second-order integrodifferential equations of motion

(LSIDEs) and nonlocal balance laws, continuity equations, or conservation laws associated with them. The formalism can be applied to classical or quantum physical systems whose relevant equations of motion are either relativistic or not. In particular it can facilitate the generation of global conservation laws for such quantities as energy and momentum.

The main result has been expressed through Theorems I–III. These relate the LSIDEs to nonlocal balance laws, continuity equations, or conservation laws for the energy-momentum (or symmetry-oriented energy-momentum) complex.

It should be noted that the formalism is applicable not only to systems of equations in field theory, but to a non-field-theoretic subset, such as those theories wherein the equations of motion have generalized coordinates as the dependent variables and either proper or coordinate time as the independent variable (e.g., in Newtonian or relativistic particle dynamics).

Furthermore, if the equations do not contain integrals, then the formalism specializes to the usual case of systems of linear second-order differential equations of motion. The energy-momentum complex for this subset will then frequently correspond to a physical system's energy and momentum instead of its energy and momentum density, respectively.

Because the LSIDEs are imbeddable in a variational statement, it is possible to alter the energy-momentum complex without changing the Euler equations. This is done by bringing elements of the so-called "null class" (p. 57 of NV) into consideration. These elements (which can also be incorporated in the variational statement) will then augment the Lagrangian chosen but thereby result in the energy-momentum complex not being unique (p. 90 of NV).

However, the null class is very useful for incorporating boundary conditions which should arise in a statement of the physical problem or, as is well known, for symmetrization of the energy-momentum complex {B26}. As a consequence of this fact, it is possible to obtain balance laws and conservation laws from nonlocal equations of motion which have rather general boundary conditions [as has been shown by Gould (1982)].

In view of the unlimited number of Lagrangians possible as auxiliary quantities in constructing the general form of the energy-momentum complexes, it was necessary to make some decision on which Lagrangians were suitable. As there is no general physical criterion for choosing one Lagrangian over another {B27}, the quantities chosen [given by equation (3.3a)] were such that they resulted in the standard expressions for the energy-momentum complex (except, perhaps for a trivial multiplicative constant) for various local equations of motion. Examples were given in the Applications section of this paper (for the local Schrödinger equation as well as for the equations corresponding to particles with spin s).

Some other cases considered (but not developed here) were as follows (using the notation $\langle N, n\rangle_m$, where N, n and m stand for the number of dependent variables, number of independent variables, and number of equations, respectively {B28}):

(a) Two coupled simple harmonic oscillators; a non field-theoretic-case of $\langle 2, 1\rangle_2$ [see equations (14-1), (14-2) and (14-23), (14-24) of Becker (1954)].

(b) The Klein–Gordon equation for a charged scalar meson interacting with an electromagnetic field and in the presence of a source (e.g., a nucleon); $\langle 1, 4\rangle_1$ {B29}.

(c) The diffusion or heat equation (both being of the same form). These, representing dissipative processes for density or heat, are of the form $\langle 1, 4\rangle_1$ {B30}.

(d) The Dirac equation for a charged particle in an electromagnetic field; $\langle 4, 4\rangle_4$ {B31}.

(e) The sourceless Maxwell equations using three complex fields, $\phi^j := E_j + iH_j$ (for $j = 1, 2, 3$); $\langle 3, 4\rangle_3$ {B32}. These are equivalent to the usual sourceless equations; $\langle 6, 4\rangle_8$. [It should be noted that the spin-s equations (4.26) of Section 4.2 include Maxwell's equations (in the presence of sources) along with their usual symmetric energy-momentum density tensor and conservation laws {B33}.]

In Section 3 it has been seen that associated with the given LSIDEs are the adjoint LSIDEs, whose dependent variables are called the *adjoint functions* (cf. Theorem I). These functions were also seen in the expression for the energy-momentum complex [cf. equation (3.3a)].

However, it is frequently possible to eliminate the adjoint functions from that complex by relating them to the given LSIDEs' dependent variables through "compatibility conditions." Examples were given in the Applications section for the nonlocal Schrödinger equation (where the adjoint function was identified as the original wave function's complex conjugate) and for the spin-s equations (where the adjoint functions were identified as the potentials $U^{\cdots}$ in the original equations of motion).

If the adjoint functions are present in a physical system's energy-momentum complex, the physical significance of those functions and of that complex appears to be an open question. For example: In the case of heat diffusing through an isotropic medium the adjoint function is said to represent the "temperature" (of an imaginary system) which *rises* with time! [p. xix of Lewins (1965) and p.113 of Moiseiwitsch (1966)]. Yet in nuclear-reactor theory one *can* ascribe physical meaning to an adjoint function (called the *importance*). [This corresponds to some detector distribution and is used to describe the probable contribution of a particle at one instant to the meter reading at a later instant, as described on p. 21 of Lewins (1965).]

The general problem of obtaining physically significant energy-momentum complexes, and their conserved quantities, from equations of motion is usually carried out within the context of the variational calculus [a recent discussion for local equations of motion was given by Lemos (1981)]. But at present all one seems capable of doing is to try and resolve this problem for each physical system considered. Indeed, even for the relatively simple case of the "pure time component" W_0^0, which corresponds to the Hamiltonian H in classical particle dynamics, one has several possibilities: H may or may not be constant in time and it also may or may not be identified as the system's total energy (Becker, 1954, p. 39).

From a mathematical perspective, however, the adjoint functions are unquestionably useful. By employing them, one can obtain necessary conditions for the existence of solutions to a system of integrodifferential equations (p. 77 of NV).

The general relation of LSIDEs to associated angular momentum complexes and their balance laws and conservation laws has been accomplished and will be reported on later. But this along with the present work does not mean that my formalism is the "best" one. In particular, an immense literature exists wherein one starts by constructing a Lagrangian for the physical system under study by using symmetry considerations. Nevertheless, it is hoped that this formalism can be easily applied to a sufficiently large number of physical systems where the equations of motion (LSIDEs in general) are known.

APPENDIX A. HOW THE MAIN RESULTS STEM FROM THE NONLOCAL VARIATIONAL CALCULUS

This section is a sketch to show how the equations of motion, energy-momentum complex, and balance laws arise out of the nonlocal calculus of variations. Further details may be found in Gould (1982) or in NV.

A1. Obtaining the Equations of Motion

The LSIDEs are obtained by employing the Lagrangian $\mathscr{L}$ defined as follows:

$$\begin{aligned}\mathscr{L} := {} & f_\Gamma \psi^\Gamma - g_\Gamma \phi^\Gamma - [a_{\Gamma\Lambda} - \tfrac{1}{2}(\partial_q a^q_{\Gamma\Lambda}) + \tfrac{1}{2}(\partial_q \partial_r a^{qr}_{\Gamma\Lambda})]\psi^\Gamma \phi^\Lambda \\ & - \tfrac{1}{2}[a^r_{\Gamma\Lambda} - (\partial_q a^{qr}_{\Gamma\Lambda})](\partial_r \phi^\Lambda)\psi^\Gamma + \tfrac{1}{2}[a^r_{\Gamma\Lambda} - (\partial_q a^{qr}_{\Gamma\Lambda})](\partial_r \psi^\Gamma)\phi^\Lambda \\ & + a^{qr}_{\Gamma\Lambda}(\partial_q \psi^\Gamma)(\partial_r \phi^\Lambda) - \tfrac{1}{2}\{\phi^\Lambda(x) k_{2\Lambda}(x; \psi^\Gamma) + [\partial_q \phi^\Lambda(x)] k^q_\Lambda(x; \psi^\Gamma) \\ & + \psi^\Gamma(x) k_{1\Gamma}(x; \phi^\Lambda)\} \end{aligned} \tag{A1}$$

where

$$k_{2\Lambda}(x;\psi^{\Gamma}) := \int_{D^*} g_{2\Lambda}[x, z, \psi^{\Gamma}(z)]\, dV(z) \tag{A1.1}$$

with

$$g_{2\Lambda}[x, z, \psi^{\Gamma}(z)] = K_{\Gamma\Lambda}(z, x)\psi^{\Gamma}(z)$$

$$k_{1\Gamma}(x;\phi^{\Lambda}) := \int_{D^*} g_{1\Gamma}[x, z, \phi^{\Lambda}(z), \partial_q\phi^{\Lambda}(z)]\, dV(z) \tag{A1.2}$$

with

$$g_{1\Gamma}[x, z, \phi^{\Lambda}(z), \partial_q\phi^{\Lambda}(z)] = K_{\Gamma\Lambda}(x, z)\phi^{\Lambda}(z) + H^q_{\Gamma\Lambda}(x, z)\partial_q\phi^{\Lambda}(z)$$

$$k^r_{\Lambda}(x;\psi^{\Gamma}) := \int_{D^*} g^r_{\Lambda}[x, z, \psi^{\Gamma}(z)]\, dV(z) \tag{A1.3}$$

with

$$g^r_{\Lambda}[x, z, \psi^{\Gamma}(z)] = H^r_{\Gamma\Lambda}(x, z)\psi^{\Gamma}(z)$$

The LSIDEs [equations (3.1)] and adjoint LSIDEs [equations (3.2)] arise from the Euler equations

$$\{\mathscr{E}|\mathscr{L}\}_{\psi^{\Sigma}}(x) = 0 \qquad \text{and} \qquad \{\mathscr{E}|\mathscr{L}\}_{\phi^{\Sigma}}(x) = 0 \tag{A1.4}$$

respectively; where

$$\{\mathscr{E}|\mathscr{L}\}_{\psi^{\Sigma}}(x) := \{e|\mathscr{L}(k_{1\Gamma}, k_{2\Lambda}, k^r_{\Lambda})\}_{\psi^{\Sigma}}(x)$$
$$+ \int_{D^*} \left(\frac{\partial \mathscr{L}}{\partial k_{1\Gamma}}(z)\{e|g^*_{1\Gamma}(z)\}_{\psi^{\Sigma}}(x) + \frac{\partial \mathscr{L}}{\partial k_{2\Lambda}}(z)\{e|g^*_{2\Lambda}(z)\}_{\psi^{\Sigma}}(x) \right.$$
$$\left. + \frac{\partial \mathscr{L}}{\partial k^r_{\Lambda}}(z)\{e|g^{r*}_{\Lambda}(z)\}_{\psi^{\Sigma}}(x) \right) dV(z) \tag{A2}$$

with the "little" Euler-Lagrange operator defined as

$$\{e|\mathscr{L}(k_{1\Gamma}, k_{2\Lambda}, k^r_{\Lambda})\}_{\psi^{\Sigma}}(x) := \frac{\partial \mathscr{L}}{\partial \psi^{\Sigma}(x)}\Big|_{k_{1\Gamma}, k_{2\Lambda}, k^r_{\Lambda}}$$
$$- \frac{\partial}{\partial x^t} \frac{\partial \mathscr{L}}{\partial[\partial_t \psi^{\Sigma}(x)]}\Big|_{k_{1\Gamma}, k_{2\Lambda}, k^r_{\Lambda}} \tag{A2.1}$$

Remarks. 1. Employing the notation defined in Remark 2 following Theorem I, one imposes the requirements that the general Lagrangian

$\mathscr{L}[x, \varphi^{\Sigma}(x), \partial_m\varphi^{\Sigma}(x), k_a(x; \varphi^{\Sigma})]$ used to specify a nonlocal functional is C^2 in its $n+2N(n+1)+Q$ arguments {B34}, where $a=1, \ldots, Q$ and

$$k_a(x; \varphi^{\Sigma}) := \int_{D^*} g_a[x, z, \varphi^{\Sigma}(z), \partial_m\varphi^{\Sigma}(z)]\, dV(z) \tag{A2.2}$$

while

$$g_a^*[x, z, \varphi^{\Sigma}(z), \partial_m\varphi^{\Sigma}(z)] := g_a[z, x, \varphi^{\Sigma}(x), \partial_m\varphi^{\Sigma}(x)]$$

2. Quantities to the right of the vertical lines, such as in equation (A2.1), are to be held constant when differentiating. Here $\partial\mathscr{L}(z)/\partial k_a$ means to first differentiate with respect to k_a and then replace x by z everywhere in the result.

A2. Obtaining the Energy-Momentum Complex

An energy-momentum complex $W_k^m(\mathscr{L})$ (with k and $m=1, \ldots, n$), based on a *general* Lagrangian [Edelen (1971*a–d*)] $\mathscr{L}$ (and related to nonlocal balance laws) can be obtained from Theorem 2.11 of NV applied to a $2N$-tuple of dependent variables φ^{Σ}. This complex is defined as the n^2 quantities

$$W_k^m(\mathscr{L}) := \frac{\partial\mathscr{L}}{\partial[\partial_m\varphi^{\Sigma}]}\partial_k\varphi^{\Sigma} + \int_{D^*}\left[w^m_{1\Gamma k}\frac{\partial\mathscr{L}}{\partial k_{1\Gamma}}(z) + w^m_{2\Lambda k}\frac{\partial\mathscr{L}}{\partial k_{2\Lambda}}(z) + w^{rm}_{\Lambda k}\frac{\partial\mathscr{L}}{\partial k^r_{\Lambda}}(z)\right] dV(z) - \delta^m_k\mathscr{L} \tag{A3}$$

where the n^2Q quantities w^m_{ak} $(a=1, \ldots, Q)$ are defined as

$$w^m_{ak} := \frac{\partial g_a^*}{\partial[\partial_m\varphi^{\Sigma}(x)]}\partial_k\varphi^{\Sigma}(x) - \delta^m_k g_a^*$$

for $a=2\Lambda$, 1Γ, or ${}^r_{\Lambda}$ and with $g_a := g_a[x, z, \varphi^{\Sigma}(z), \partial_m\varphi^{\Sigma}(z)]$ and $k_a := k_a(x; \varphi^{\Sigma})$ [following the notation of equation (A2.2)].

Substituting the Lagrangian of (A1) into the energy-momentum complex (A3) yields the specific complex of equations (3.3).

A3. The Symmetry-Oriented Energy-Momentum Complex

Using a general Lagrangian $\mathscr{L}$ (mentioned above), the "symmetry-oriented" energy-momentum complex will be defined as

$$\Theta^{\alpha\beta}(\mathscr{L}) := \mathscr{W}^{\alpha\beta}(\mathscr{L}) - \partial_\mu G^{\alpha\beta\mu}(\mathscr{L}) \tag{A4}$$

where

$$\mathcal{W}^{\alpha\beta}(\mathscr{L}) := \frac{1}{2} W^{\alpha\beta}(\mathscr{L})$$

with $W^{\alpha\beta}(\mathscr{L})$ obtained from equation (A3).

The quantities making up $G^{\alpha\beta\nu}(\mathscr{L})$ are defined as

$$G^{\alpha\beta\mu}(\mathscr{L}) := \tfrac{1}{2}[S^{\beta\mu\alpha}(\mathscr{L}) + S^{\alpha\mu\beta}(\mathscr{L}) - S^{\alpha\beta\mu}(\mathscr{L})] \tag{A4.1}$$

where

$$S^{\mu}_{\alpha\beta}(\mathscr{L}) := \frac{1}{4} \frac{\partial \mathscr{L}}{\partial[\partial_\mu \varphi^{\Sigma}(x)]} [\tilde{M}_{\alpha\beta}]^{\Sigma}_{\Omega} \varphi^{\Omega} \tag{A4.2}$$

with $[\tilde{M}_{\alpha\beta}]^{\Sigma}_{\Omega}$ given by equations (3.7). Antisymmetry of $M_{\alpha\beta}$ [from equation (3.7a)] leads to $S^{\mu}_{\alpha\beta}(\mathscr{L}) = -S^{\mu}_{\beta\alpha}(\mathscr{L})$ and thus $G^{\alpha\beta\mu}(\mathscr{L}) = -G^{\alpha\mu\beta}(\mathscr{L})$, using definitions (A4.2) and (A4.1), respectively.

Substituting the Lagrangian of (A1) into equation (A4.2) yields equation (3.6).

A4. Divergence of the Energy-Momentum Complexes

Again using a general Lagrangian $\mathscr{L}$, with associated Euler equations (A1.4) and the quantities $W^{m}_{k}(\mathscr{L})$ specified in equation (A3), an immediate extension (Gould, 1982) of Theorem 2.11 of NV gives the nonlocal balance laws

$$\partial_m W^{m}_{k}(\mathscr{L}) := -\partial^{*}_{k}\mathscr{L} - \int_{D^*} \left[\frac{\partial \mathscr{L}}{\partial k_a}(z) \partial^{*}_{k} g^{*}_{a} + \frac{\partial \mathscr{L}}{\partial k_\alpha} \partial_k g_a \right] dV(z) \tag{A5}$$

where

$$\partial^{*}_{k} g^{*}_{a} := \partial_k g^{*}_{a}|_{\varphi^{\Sigma}(x),\, \partial_m \varphi^{\Sigma}(x),\, z}, \qquad \partial^{*}_{k}\mathscr{L} := \partial_k \mathscr{L}|_{\varphi^{\Sigma}(x),\, \partial_m \varphi^{\Sigma}(x),\, k_a}$$

and $\partial_k := \partial/\partial x^k$. Furthermore,

$$\partial_\beta \Theta^{\alpha\beta}(\mathscr{L}) = \partial_\beta \mathcal{W}^{\alpha\beta}(\mathscr{L})$$

Substituting the Lagrangian of (A1) into equation (A5) then yields equation (3.8).

APPENDIX B. ADDITIONAL NOTES

1. Each of these laws is to be widely construed. Thus, it is not necessary, for example, that a conservation law be derivable from an

equation of motion or vice versa. Furthermore, the term "equation of motion," is not restricted to, say, Newton's second law, but can also refer, for example, to Schrödinger's equation or to a system of linear stochastic integrodifferential equations. [A recent illustration of the latter is by Volkov and Pokrovsky (1983).]

2. The reader should note that there is no universal agreement concerning the form of a conservation law. Many authors refer to the former type of equation as a "continuity equation" as well as a (local) "conservation law" (e.g., Anderson, 1967, pp. 93 and 94; Misner *et al.*, 1973, pp. 132 and 152). Other authors refer to the constancy in time for a space-integrated quantity, such as W, as a "conservation law" [see, e.g., Goldstein (1980), p. 557; Soper (1976), pp. 30–33, where the constancy for different reference frames is discussed.]

3. This should not be taken to imply that such an imbedding can always be accomplished (even though it is possible in many cases of physical interest). A discussion of this issue for differential equations of motion can be found in Havas (1957, 1973).

4. It may be objected as this point that if one converted the integrodifferential equations to corresponding differential equations with appropriate boundary conditions, then a nonlocal variational calculus would be obviated. However, (i) while it is true that some nonlocal equations have local counterparts (such as the Lippmann–Schwinger equation for the usual Schrödinger equation) this is not always the case. "There are indeed integrodifferential equations that are not equivalent to any system of differential equations of finite order," as mentioned by Edelen (in Eringen, 1976, Vol. IV, p. 76). (ii) When it is possible to go from a nonlocal to a local equation wherein the latter has higher-order derivatives, the variational imbedding would generally impose more stringent continuity requirements on the Lagrangian and may create serious problems for the physical interpretation of higher-order boundary conditions. (iii) If one starts out with nonlocal equations, it just complicates matters to try and force them into a local form. It also tends to obscure the nonlocal aspect of the original equations as well as this same aspect of the balance laws, continuity equations, or conserved quantities associated with them.

5. Here it is significant to note that "classical singularities in the expressions of the stress field and stored energy are found not present in the nonlocal model" (Eringen, 1977).

6. Eringen (1972)—the first continuum theory of nonlocal fluid mechanics. Also see Edelden (1975).

7. Eringen (1973)—the first continuum theory of nonlocal electromagnetism. Hajdo and Eringen (1979) give an application to electromagnetic dispersion.

8. Fokker, (1929) first presented the nonlocal variational principle for electrodynamics. See also Havas [in Bunge, 1971, p. 31].

9. Applications of a nonlocal potential in Schrödinger's equation can be found in Silver and Austern (1980), and more recently, employing a quark model of the deuteron, in Williams *et al.* (1982). Also, although there has been some dispute concerning the cause of the Aharonov-Bohm effect (Roy, 1980, and references therein), some authors have attributed the cause to nonlocal effects [see, e.g., the implications of results by Purcell and Henneberger (1978)].

10. Rzewuski (1958*a*,*b*) also has a lengthy discussion of causality problems.

11. See the volumes edited by Eringen (1976); references start on p. 201 (from Edelen) and on p. 265 (from Eringen).

12. The first seven articles can be found in essence on pp. 1-95 and in the Appendix of NV.

13. Edelen refers to these quantities as "momentum-energy complexes" (p. 83 of NV) for historical reasons; but, as will be shown, they do not necessarily represent the energy-momentum or energy-momentum density.

14. (a) According to comments by Eddington (1924) concerning the "action" (whose integrand is the Lagrangian): "From its first introduction action has always been looked upon as something whose sole *raison d'être* is to be varied— and, moreover, *varied in such a way as to defy the laws of nature*!" (b) In view of the widespread belief in and use of Noether's theorem it must be stressed: That theorem has been generalized by Edelen (1971*a*-*d*). Certain results from his treatment, viz., the general form of the energy-momentum-complex functions and their related divergences [also discussed in Edelen (1969*a*)] have been used and are briefly discussed in Appendix A of this paper.

15. It should be noted that to arrive at the formalism a generalized Lagrangian was employed (as shown in Appendix A). It was chosen because it was found to apply to a large number of physical systems. [One may, of course, choose the other Lagrangians from an equivalence class of null Lagrangians; as done, e.g., in Gould (1982) and in NV.]

16. Supported in part by NSF Grant PHY 77-28356.

17. A general theory of angular momentum for physical situations described by a system of LSIDEs has been worked out in Gould (1982) and will be further developed in a future paper.

18. A nice example of a *local* momentum-energy complex (corresponding to the ${}^{L}W_{k}^{m}$ of this paper) applicable to elasticity, arising out of a Noetherian treatment and found in "transversality conditions," can be seen in Edelen (1981).

19. An elementary example is given by Byron and Fuller (1970), equation (8.64). For more advanced applications see Foldy and Lock (1979) or Silver and Austern (1980). A recent example of a nonlocal Schrödinger equation appearing in QCD for a quark model of the deuteron is by Williams *et al.* (1982).

20. A brief summary of the following procedure appeared in Gould (1986).

21. (a) Cf. equation (8.65) of Byron and Fuller (1970). [One could have arrived at equation (4.8) from the definition of the Hermiticity of H along with Green's theorem, but that would have required more work than the present method.] (b) For

$$U(x, z) = \tfrac{1}{2}[U(x)\delta(x-z) + U(z)\delta(z-x)]$$

satisfying equation (4.8) where $\delta(\cdots)$ is the Dirac delta function], the nonlocal term in equation (4.1) reduces, for real U, to

$$\int_{D^*} U(x, z)\phi(z)\, dV(z) = U(x)\phi(x)$$

the usual term involving the potential in the local Schrödinger equation.

22. Using equation (4.8), it follows that equation (4.14) becomes

$$W_4^4 = (\hbar^2/2M)(\nabla\phi)\cdot(\nabla\bar{\phi}) + 2\Re[\bar{\phi}(x)\int_{D^*} U(x, z)\phi(z)\, dV(z)]$$

which does not reduce to the usual local expression

$$W_4^4 = (\hbar^2/2M)(\nabla\phi)\cdot(\nabla\bar{\phi}) + U(x)\bar{\phi}(x)\phi(x)$$

if $U(\cdot) = (1/2)[\cdot]$\{B21\} is used for real U (and does not lead to a conservation law) even though one does recover the local Schrödinger equation for this U. This does not mean there is an inconsistency, for it is a situation that has also occurred in other nonlocal field theories [see, e.g., Rzewuski (1958*b*), p. 245 and p. 243 below equation (8.4)] and may be due to: (a) the accumulation of certain nonlocal effects being essentially nonreducible to local ones, or (b) the nonrigorous use of the δ function (e.g., Grotch *et al.*, 1982).

23. For the case $k = 1$ there (i.e., for a single particle). But note that: (a) the rhs of equation (4.24) is *not* zero, when the equation is put in component form, as it is in the more specialized case of his equation (II S), and (b) the rhs of equation (4.25) is not zero as it is in equation (I S)

of his paper (which is expected, since he uses a time-independent potential energy).

24. See, for example, Havas (1959) and references therein, especially Fierz (1939); for the charge-symmetric case [also worked out in Gould (1982)] see Le Couteur (1949).

25. If it was known that there were no sources (i.e., $J^{\cdots}\equiv 0$), then one could have arrived at $\partial_\beta\theta^{\alpha\beta}=0$ more rapidly because $s^{\mu}_{\alpha\beta}$ and $g^{\alpha\beta\mu}$ would not contribute; i.e., in this case $\Theta^{\alpha\beta}\equiv\theta^{\alpha\beta}$.

26. In fact the freedom to choose an appropriate null-class element was used in constructing the general symmetry-oriented energy-momentum complex of Section 3.

27. This is true even for conservative systems as shown, e.g., by Havas (1957) and Hojman and Urrutia (1982).

28. This notation was suggested by the late Prof. Henri Amar (Temple University).

29. In particular, when $\rho=0$ for the source density then the $\mathscr{W}_0^0$ component is equivalent to equation (3.258) on p. 123 of Moiseiwitsch (1966).

30. The $\mathscr{W}_4^4$ component is essentially identical to equation (3.211) of Moiseiwitsch (1966).

31. The integrated W_0^0 component is essentially identical to $\mathscr{H}$ of equation (3.287) of Moiseiwitsch (1966).

32. The appropriate W_k^m coincide essentially with those of equation (23b) of Good (1957) but are easier to obtain than by using his method.

33. The result is hardly surprising if one knows in advance that the photon corresponds to a massless spin-1 field. See Soper (1976) for the Maxwell results.

34. It should be noted that such a Lagrangian, more general than that of equation (A1), can imbed nonlocal Euler equations which are also nonlinear. Furthermore, the variational imbedding does not require φ^{Σ} to be C^2. [Conditions under which the C^2 requirement need not hold can be found in Gould (1982) and in NV.]

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